

**Relativistic quantum mechanics  
of a neutral two-body system  
in a constant magnetic field**

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*A (globally) neutral two-body system is supposed to obey a pair of coupled Klein-Gordon equations in a constant homogeneous magnetic field. Considering eigenstates of the pseudomomentum four-vector, we reduce these equations to a three-dimensional eigenvalue problem.*

*The frame adapted to pseudomomentum has in general a nonvanishing velocity with respect to the frames where the field is purely magnetic. This velocity plays a crucial role in the occurrence of motional terms; these terms are taken into account within a manifestly covariant framework.*

*Perturbation theory is available when the mutual interaction does not depend on the total energy; a weak-field-slow-motion approximation is more specially tractable.*

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## 1. INTRODUCTION.

From a relativistic point of view, the constant magnetic field has this peculiarity that it does not correspond to a unique laboratory frame. When a constant homogeneous electromagnetic field is seen as purely magnetic in some frame (conventionally referred to as laboratory frame), such a frame cannot be unique [1]. The directions eligible for the time axis of a lab frame span a 2-dimensional hyperbolic space ( $E_L$ ) (longitudinal space). Energy, defined as time component of the momentum, is conserved but it is affected by this ambiguity about the lab frame. Therefore a manifestly covariant treatment is specially relevant.

Another important feature of the constant magnetic field is the conservation of pseudomomentum, which can be checked in several cases of interest. In these cases, the true momentum (associated with translations in configuration space) is not fully conserved, because application of a magnetic field spoils translation invariance in the Klein-Gordon operator, but it turns out that the so-called *twisted translations* [2] operating in phase space, preserve the dynamics, which results in the occurrence of a conserved vector [2-5] which reduces to ordinary momentum in the no-field limit.

At least this statement is true for  $n$  charges in the Galilean theory, supposing that the mutual interaction between two charges corresponds to a central force [2].

It is also true for one-body and two-body relativistic systems, under rather general assumptions. Checking the one-body case is straightforward: the four-velocity being  $p - eA$ , and the gauge being  $A = \frac{1}{2}q \cdot F$ , it is a mere exercise to verify that  $C = p + eA$  commutes with the Klein-Gordon operator.

For scalar particles, the *two-body problem* can be posed in terms of a pair of coupled wave equations [6,7].

In the absence of external field, the dynamics of the system has all the Poincaré symmetries, in particular it is invariant under ordinary translations; as a result total momentum  $P$  is conserved. Then, imposing sharp values to its components and after elimination of the relative time, one is left with a reduced wave equation involving three degrees of freedom only. At least when the mutual interaction potential does not depend on the total energy [8], this reduced equation can be seen as a standard eigenvalue problem (in terms of the energy of relative motion). As the constituent masses have been fixed from the outset, solving this equation yields a spectrum (generally with a discrete part). Admissible values of  $P^2$  are correspondingly selected.

When an external electromagnetic field is applied to the system, a first problem consists in keeping the wave equations consistent with one another. When the field can be seen as purely magnetic and constant (in space and time) for some observer, the compatibility requirement, combined with symmetry conditions and the demand of reasonable limits when mutual (resp. external) coupling is turned off, is satisfied in closed form by an ansatz [1][9][10], provided the that the mutual interaction term is known in the isolated system.

Now the customary momentum  $P$  is not any more a constant of the motion. But we have previously proved this result [1,10]: for a large class of mutual interactions (sufficient for all practical purposes) the *total pseudomomentum of (globally) neutral systems* (that is  $e_1 = -e_2 = e$ ) is actually conserved. Moreover the four components of this vector commute among themselves. This result seems to admit an extension to the case of an arbitrary constant electromagnetic field [10], but only the pure electric or magnetic cases lead to a rigorous statement and to wave equations written in closed form, after a suitable transformation [11].

It is possible to take advantage of pseudomomentum conservation in order to reduce the degrees of freedom, in a way which parallels the customary separation of center-of-mass variables usually performed in isolated systems. In addition, a further degree of freedom can be separated out. *In the magnetic case* (considered throughout this paper), this degree of freedom is a timelike variable identified as relative time in one of the possible lab frames.

This situation provides a description of neutral bound states in a magnetic field. Indeed all the components of pseudomomentum can be simultaneously diagonalized. In a generalized sense, the motion of the system as a whole is separated out, and inner motion can be considered; in the same spirit as in the nonrelativistic theory, it is natural to associate a spectrum to this (pseudo) relative motion.

Solving the reduced eigenvalue equation will determine admissible values for the square of pseudomomentum; having bound states in mind, we shall be more specially interested in the discrete spectrum.

Since pseudomomentum is the generalization of momentum in the presence of field, its square plays the role of an *effective squared mass* and should be observable. We are led to investigate how much this quantity is shifted by the field from the total squared mass we would have obtained for the isolated system.

In a suitable representation, we shall explicitly write down and reduce the wave equations.

As the reduced equation is a three-dimensional problem, it is tempting to apply the standard methods of perturbation theory. But, even in the simple case where the unperturbed motion is ruled by a mutual interaction term which does not depend on  $P^2$ , a complication arises due to the presence of external field: the reduced equation nonlinearly depends on the total energy. As a result, it is not an eigenvalue equation in the usual sense. Fortunately, a spectral theory for *energy-dependent perturbations of a standard eigenvalue equation* has been already presented in the literature [12].

Let us now say a few words about the geometric elements corresponding to a constant magnetic field in the presence of a constant timelike vector  $k$ , eigenvalue of pseudomomentum.

A constant magnetic field  $F^{\mu\nu}$  provides a *unique and relativistically invariant* decomposition [1] of any four-vector  $\xi$  into longitudinal and transverse parts,  $\xi = \xi_L + \xi_T$ .

In particular  $k = k_L + k_T$ . Assuming that  $k$  is timelike, it is clear that  $k_T$  can accidentally vanish whereas  $k_L$  never does. In fact  $k_L$  is always timelike. Since  $k_L \in (E_L)$  it can be considered as defining a preferred lab frame, we shall refer to it as the *special lab frame*. Let us emphasize that this preferred lab frame also depends on the state of motion of the system (in contradistinction to the unique lab frame associated with an inhomogeneous field).

It is noteworthy that  $C_L^\alpha = P_L^\alpha$  and therefore  $|k_L|$  is nothing but the energy seen in the special lab frame.

In general the frame adapted to  $k$  (which differs from  $k_L$ ) cannot be a lab frame. In view of results about pseudomomentum in the Galilean theory, it is reasonable to admit that  $k$  carries information about the motion of the system as a whole. So we call *pseudo-rest frame* the frame adapted to  $k$ .

With respect to the special lab frame, it has a velocity characterized by the *motional parameter*  $\epsilon = \frac{|k_T|}{|k_L|}$ .

In the particular case where  $k_T = 0$ , then  $k_L$  and  $k$  coincide and  $\epsilon$  vanishes; there exists a frame adapted to  $k$  where  $F$  is purely magnetic and the subsequent developments get drastically simplified. Otherwise, terms involving the contraction  $k \cdot F$  arise. We notice that  $(k \cdot F)^\alpha = |k|E^\alpha$  where  $E^\alpha$  is the electric field "seen" by an observer moving with momentum  $k^\alpha$  (*motional electric field*).

This paper is organized as follows. In next section we recall the ansatz which provides explicit equations of motion, and simplify a quantity  $\widehat{Z}$  which was an essential ingredient in the formulas of refs.[1, 10]. In Section 3, several expressions given in compact form in that previous work are explicitly developed for applications. Section 4 is devoted to the three-dimensional reduction, with emphasis on the role of relative energy. Various terms present in the reduced equation are discussed and ordered according to the powers of field strength and motional parameter. In Section 5 we consider normalization and the possibility of a reliable perturbation treatment. Last Section is devoted to a few concluding remarks.

## 2. BASIC EQUATIONS

A system of scalar particles can be described by a pair of coupled Klein-Gordon equations  $H_a \Psi = \frac{1}{2}m_a^2 \Psi$ ,  $a, b = 1, 2$  where  $\Psi$  has two arguments  $q_1, q_2$  running in spacetime. We cover all cases of practical interest assuming that

$$H_a = K_a + V$$

In this formula  $2K_a$  is the squared-mass operator for particle  $a$  alone in the magnetic field, and  $V$  is a suitable modification of the term  $V^{(0)}$  which describes the mutual interaction in the absence of external field; more generally, the label (0) refers to the no-field limit of any quantity.

In order to be more specific we separate canonical variables in two classes as follows.

$$\begin{aligned} P &= p_1 + p_2, & Q &= \frac{1}{2}(q_1 + q_2) \\ z &= q_1 - q_2, & y &= \frac{1}{2}(p_1 - p_2) \end{aligned}$$

This provides two sets of standard commutation relations. Useful quantities are

$$\tilde{y}^2 = y^2 - (y \cdot P)^2 / P^2$$

$$\tilde{z}^2 = z^2 - (z \cdot P)^2 / P^2$$

The latter is an essential ingredient of mutual interactions; but in order to avoid denominators in calculations, it is convenient to employ

$$Z = z^2 P^2 - (z \cdot P)^2 \quad (2.1)$$

We shall assume that

$$V^{(0)} = f(Z, P^2, y \cdot P) \quad (2.2)$$

Since our system is globally neutral, it turns out that pseudomomentum is

$$C = P + \frac{e}{2} z \cdot F$$

In canonical variables we can separate the transverse pieces from the longitudinal ones, for instance  $P = P_t + P_L$ , etc. Transverse and longitudinal variables mutually commute.

Compatibility requires that  $[K_1 - K_2, V] = 0$ . Adapting Bijtebier's method [9] to the peculiarities of the constant magnetic field, we have performed a canonical transformation say

$$\Psi' = \exp(iB) \Psi, \quad \mathcal{O}' = \exp(iB) \mathcal{O} \exp(-iB) \quad (2.4)$$

where  $\mathcal{O}$  is any operator.  $B$  was chosen [1,13] such that transformation (2.4) yields

$$K'_1 - K'_2 = y_L \cdot P_L \quad (2.5)$$

thus compatibility is satisfied through the ansatz [1]

$$V' = f(\hat{Z}, P^2, y_L \cdot P_L) \quad (2.6)$$

where  $\hat{Z} = Z'_{(0)} = (Z')_{F=0}$  (it turns out that  $\hat{Z}$  commutes with  $y_L \cdot P_L$ ). The explicit form of  $\hat{Z}$  was calculated in ref.[1]

$$\hat{Z} = Z + 2(P_L^2 z \cdot P - P^2 z_L \cdot P_L)L + P_t^2 P_L^2 L^2 \quad (2.7)$$

where the scalar  $L$  is defined as  $L = L \cdot z$  in terms of the the four-vector [14]

$$L^\alpha = \frac{P_L^\alpha}{(P_L)^2} \quad (2.8)$$

As it stands, formula (2.7) is of poor practical interest. It is essential to observe [1] that  $\hat{Z}$  commutes with  $y_L \cdot P_L$ . Let us transform (2.7) in order to render this property manifest. First we split  $z$  as the sum of  $z_L$  and  $z_T$  in  $Z$ , hence

$$Z = (z_T^2 + z_L^2)P^2 - (z_T \cdot P)^2 - (z_L \cdot P)^2 - 2(z_T \cdot P)(z_L \cdot P) \quad (2.9)$$

Develop (2.7) and perform elementary manipulations using (2.9). We get

$$\hat{Z} = Z + 2P_L^2(z_T \cdot P)L + 2P_L^2(z_L \cdot P)L - 2\frac{P^2}{P_L^2}(z_L \cdot P)^2 + \frac{P_t^2}{P_L^2}(z \cdot P_L)^2$$

Using (2.9) again we obtain

$$\hat{Z} = Z + 2(z_T \cdot P)(z_L \cdot P) + 2(z_L \cdot P)^2 - 2\frac{P^2}{P_L^2}(z_L \cdot P)^2 + \frac{P_t^2}{P_L^2}(z_L \cdot P)^2$$

That is

$$\hat{Z} = Z + 2(z_T \cdot P)(z_L \cdot P) - (z_L \cdot P)^2 \frac{P_t^2}{P_L^2}$$

Using (2.9) we notice cancellation of the terms proportional to  $(z_T \cdot P)(z_L \cdot P)$  and we can write

$$\widehat{Z} = z_T^2 P^2 + z_L^2 P^2 - (z_T \cdot P)^2 - (z_L \cdot P)^2 (1 + \frac{P_T^2}{P_L^2})$$

That is to say

$$\widehat{Z} = z_T^2 P^2 - (z_T \cdot P)^2 + P^2 (z_L^2 - \frac{(z_L \cdot P_L)^2}{P_L^2}) \quad (2.10)$$

It is convenient to define the projector "orthogonal" to  $P_L$ , say

$$\Omega_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} - \frac{P_L^{\alpha} P_L^{\beta}}{P_L^2} \quad (2.11)$$

because we can write

$$z_L^2 - \frac{(z_L \cdot P_L)^2}{P_L^2} = (\Omega z_L)^2 \quad (2.12)$$

and we easily check that  $(\Omega z)^{\alpha}$  commutes with  $(y_L \cdot P_L)$ . So we finally have

$$\widehat{Z} = z_T^2 P^2 - (z_T \cdot P)^2 + (\Omega z_L)^2 P^2 \quad (2.13)$$

which justifies the claim that  $\widehat{Z}$  commutes with  $y_L \cdot P_L$ . Here we notice that  $\Omega z_T = z_T$  and finally obtain

$$\widehat{Z} = (\Omega z)^2 P^2 - (z_T \cdot P)^2 \quad (2.14)$$

This simplification of (2.7) renders the ansatz more tractable.

Remark: Formula (2.14) was derived without specifying the respective dimensions of the longitudinal and transverse spaces. It is valid also in the case considered in ref.[9], but in that case  $\Omega z$  reduces to  $z_T$ , which makes  $\Omega z_L$  to vanish.

It is convenient to replace the basic wave equations by their sum and difference, setting

$$\mu = \frac{1}{2}(m_1^2 + m_2^2), \quad \nu = \frac{1}{2}(m_1^2 - m_2^2)$$

After transformation (2.4), and in view of (2.6) we need the expression of  $K'_1 + K'_2$ . Equations (3.36) of Ref.[1] yield in the present notation

$$K'_1 + K'_2 = K_1 + K_2 - 2T(L \cdot y) + T^2(L \cdot L) \quad (2.15)$$

where

$$T = K_1 - K_2 - y_L \cdot P_L \quad (2.16)$$

and the four-vector  $L$  is given by (2.8). We notice that

$$L \cdot L = \frac{1}{(P_L)^2}, \quad L \cdot y = \frac{y_L \cdot P_L}{P_L^2} \quad (2.17)$$

For neutral systems, a further transformation inspired by the work of Grotch and Hegstrom [4] permits to get rid of the  $Q$  variables. We introduce a new wave function  $\Psi'' = (\exp i\Gamma)\Psi'$  with the help of the unitary transformation generated by

$$\Gamma = \frac{e}{2}(z \cdot F \cdot Q) \quad (2.18)$$

We set

$$\mathcal{O}^{\#} = \exp(i\Gamma) \mathcal{O} \exp(-i\Gamma) \quad \mathcal{O}'' = (\mathcal{O}')^{\#} \quad \forall \mathcal{O} \quad (2.19)$$

By (2.19), pseudomomentum is transformed to  $P^\alpha$ . Since  $y_L \cdot P_L$  and  $P_L^\alpha$  commute with  $\Gamma$ , they are not affected by transformation (2.19), thus finally

$$H_1'' - H_2'' = K_1'' - K_2'' = y_L \cdot P_L \quad (2.20)$$

Taking (2.20) into account, consider sum and difference of the wave equations.

$$(H_1'' + H_2'')\Psi'' = \mu\Psi'' \quad (2.21)$$

$$y_L \cdot k_L \Psi'' = \nu\Psi'' \quad (2.22)$$

Transformation (2.19) ensures that  $H_1'' + H_2''$  does not depend on the external variable  $Q$ . Moreover  $H_1'' + H_2''$  commutes with  $y_L \cdot P_L$ ; therefore it should not involve the operator  $z_L \cdot P_L$  conjugate to it. We shall explicitly check this point in Section 4.

From now on we demand that  $\Psi''$  be eigenstate of pseudomomentum with a timelike four-vector  $k^\alpha$  as eigenvalue. Combining this requirement with (1.22) we obtain

$$\Psi'' = \exp(ik \cdot Q) \exp(i\nu \frac{z_L \cdot k_L}{|k_L|}) \phi \quad (2.23)$$

where  $\phi$  depends on  $z$ , but only through its projection orthogonal to  $k_L$ , and additionally depends on  $k$  and on  $\nu$  as parameters. In other words  $\phi = \phi(\nu, k, \varpi z)$  where we define  $\varpi$  as the tensor which projects any vector on the 3-plane orthogonal to  $k_L$ . The linear space of such functions includes as a subspace the Hilbert space

$$L^2(k_L) = L^2(\mathbf{R}^3, d^3\varpi z) \quad (2.24)$$

characterized by convergence of the triple integral  $\int \phi^* \phi d^3\varpi z$  which simply reads  $\int \phi^* \phi d^3\mathbf{z}$  in any frame adapted to  $k_L$ . We use the subscript  $k$  for typographical simplicity; it refers to the *vector*  $k$ . In fact, the Hilbert space defined above depends on  $k$  only through the *direction* of  $k_L$  (it does not depend on  $k_L^2$ ).

Remark: There is no explicit contribution of external field to equations (2.22) (2.23). In contradistinction, equation (2.21) remains sensitive to the presence of external field.

### 3. EXPLICIT FORMULAS

Our goal is to discuss the system (2.21)(2.22). In order to have an explicit expression for (2.21), let us first calculate  $H_1'' + H_2''$ . According to the notation set in (2.19) and to the definition of  $\Psi''$  we can write

$$H_1'' + H_2'' = (K_1' + K_2')^\sharp + 2V'^\sharp = K_1'' + K_2'' + 2V'' \quad (3.1)$$

We have to transform  $(K_1' + K_2')$  and  $V'$  as indicated in (2.19). But  $V'$  is given by (2.6). According to (3.1), we need to transform formulas (2.15) and (2.6). Notice that the transformation generated by  $\Gamma$  always can be carried out explicitly. It leaves  $q_1, q_2$  unchanged whereas

$$p_1^\sharp = p_1 - \frac{e}{2} F \cdot q_2 \quad p_2^\sharp = p_2 + \frac{e}{2} F \cdot q_1 \quad (3.2)$$

Hence

$$P^\sharp = P + \frac{e}{2} F \cdot z \quad (3.3)$$

$$y^\sharp = y - \frac{e}{2} F \cdot Q \quad (3.4)$$

Notice that  $F \cdot z$  has only transverse components and they depend on the transverse variables  $z_T$  only. Therefore we can write

$$P_L^\sharp = P_L \quad (3.5)$$

Eq (3.3) also implies

$$P_L^{\sharp 2} = P^2 + eP \cdot F \cdot z + \frac{e^2}{4} (F \cdot z)^2 \quad (3.6)$$

$$(P_T^\sharp)^2 = P_T^2 + e(P \cdot F \cdot z) + \frac{e^2}{4}(F \cdot z)^2 \quad (3.7)$$

These formulas will be useful later on.

Straightforward calculation made in ref [1] yields

$$K_a^\sharp = \frac{1}{2}(p_a - \frac{e}{2}z \cdot F)^2 \quad (3.8)$$

Obviously  $z^\sharp = z$ , and  $P^\sharp$  was given in (3.3) above. Explicit development gives

$$2K_1^\sharp = p_1^2 - \frac{e}{2}p_{1\alpha}z_\sigma F^{\sigma\alpha} - \frac{e}{2}z_\sigma F^{\sigma\alpha}p_{1\alpha} + \frac{e^2}{4}(z \cdot F)^2 \quad (3.9)$$

$$2K_2^\sharp = p_2^2 - \frac{e}{2}p_{2\alpha}z_\sigma F^{\sigma\alpha} - \frac{e}{2}z_\sigma F^{\sigma\alpha}p_{2\alpha} + \frac{e^2}{4}(z \cdot F)^2 \quad (3.10)$$

We observe that all terms quadratic in the charge disappear from the difference; we get

$$2(K_1 - K_2)^\sharp = p_1^2 - p_2^2 - e(y_\alpha z_\sigma + z_\sigma y_\alpha)F^{\sigma\alpha}$$

As  $[z_\sigma, y_\alpha]$  is symmetric as a tensor, the order of  $y, z$  is immaterial when multiplied by the skew-symmetric  $F$ . Recall that  $p_1^2 - p_2^2 = y \cdot P$ , hence the useful formula

$$(K_1 - K_2)^\sharp = y \cdot P - 2e z \cdot F \cdot y \quad (3.11)$$

But our goal was to transform eq (2.15). We first calculate  $(K_1 + K_2)^\sharp$ .

Notice that  $P \cdot (z \cdot F) = (z \cdot F) \cdot P = z \cdot F \cdot P$ , hence

$$2(K_1 + K_2)^\sharp = p_1^2 + p_2^2 - ez \cdot F \cdot P + \frac{e^2}{2}(z \cdot F)^2$$

But  $p_1^2 + p_2^2 \equiv P^2/2 + 2y^2$  thus

$$(K_1 + K_2)^\sharp = \frac{P^2}{4} + y^2 - \frac{e}{2}z \cdot F \cdot P + \frac{e^2}{4}(z \cdot F)^2 \quad (3.12)$$

Then we have to transform  $T$ ,  $L \cdot y$  and  $L \cdot L$ . Take (2.16) into account and remember (3.11). (we know that  $y_L \cdot P_L$  and  $P_L$  are not affected by  $\sharp$ ). We get

$$T^\sharp = y_T \cdot P_T - 2ez \cdot F \cdot y \quad (3.13)$$

And finally after a glance at (2.17) we observe that  $L \cdot y$  and  $L \cdot L$  are unchanged in the transformation generated by  $\Gamma$ .

Now we apply transformation (2.19) to (2.15), taking (3.12)(3.13) into account. It gives

$$\begin{aligned} K_1'' + K_2'' = \\ \frac{P^2}{4} + y^2 - \frac{e}{2}z \cdot F \cdot P + \frac{e^2}{4}(z \cdot F)^2 - 2(y_T \cdot P_T - 2ez \cdot F \cdot y)L \cdot y + (y_T \cdot P_T - 2ez \cdot F \cdot y)^2 L \cdot L \end{aligned} \quad (3.14)$$

We know that  $2V''$  must be added to this expression in order to obtain  $H_1'' + H_2''$ . But in (2.18)  $F^{\mu\nu}$  is purely transversal, therefore  $(y_L \cdot P_L)^\sharp = y \cdot P_L$ . We have by (2.6)

$$V'' = f(\widehat{Z}^\sharp, P^\sharp, y_L \cdot P_L) \quad (3.15)$$

where  $P^{\sharp 2}$  is as in (3.6) and we must compute  $\widehat{Z}^{\sharp}$  from (2.13) by help of (3.3)(3.5). We make the convention that  $\widehat{Z}^{\sharp} = (\widehat{Z})^{\sharp}$  (and not the reverse).

Then we apply the transformation (2.19) to eq. (2.13). Inspection of (3.5) shows that  $(\Omega z_L)^2$  is not affected by the transformation. We notice that  $z_T \cdot P^{\sharp} = z_T \cdot P$  because,  $F$  being purely transverse,  $z_T \cdot F \cdot z$  identically vanishes. Thus

$$\widehat{Z}^{\sharp} = P^{\sharp 2}(\Omega z)^2 - (z_T \cdot P)^2 \quad (3.16)$$

Now, eqs (3.1)(3.14)(3.15) supplemented with (3.6) and (3.16) furnish the complete expression of  $H_1'' + H_2''$ .

#### 4. THREE-DIMENSIONAL REDUCTION

##### 4.1 The reduced wave equation for isolated systems

Our presentation of the three-dimensional reduction will be more transparent if we first remind how it is usually carried out (and how it leads to a spectrum for  $P^2$ ) when the system is free of external forces.

In this subsection we drop the superscripts (0) referring to an isolated system. In the absence of external field, the squared-mass operators can be written  $2H_a = p_a^2 + 2V$ . A little of elementary algebra provides

$$H_1 + H_2 = \frac{P^2}{4} + y^2 + 2V \quad (4.1)$$

$$H_1 - H_2 = y \cdot P \quad (4.2)$$

and the wave equations take on the form

$$\left( \frac{P^2}{4} + y^2 + 2V \right) \Psi = \mu \Psi \quad (4.3)$$

$$(y \cdot P) \Psi = \nu \Psi \quad (4.4)$$

Introducing  $\tilde{y}^2$  as defined in Section 2 we get

$$H_1 + H_2 = \frac{P^2}{4} + \frac{(y \cdot P)^2}{P^2} + \tilde{y}^2 + 2V \quad (4.5)$$

and the wave equation (4.3) now reads

$$\left( \frac{P^2}{4} + \frac{(y \cdot P)^2}{P^2} + \tilde{y}^2 + 2V \right) \Psi = \mu \Psi \quad (4.6)$$

Introduce the operator [6, 15]

$$N = \tilde{y}^2 + 2V \quad (4.7)$$

equation (4.6) takes on the form

$$\left( \frac{P^2}{4} + \frac{(y \cdot P)^2}{P^2} + N \right) \Psi = \mu \Psi \quad (4.8)$$

It is noteworthy that  $-N$  is intimately related with the energy of relative motion (see the nonrelativistic limit  $\lambda \ll \mu$ ). Taking (4.1)(4.2) into account it is easy to check the identity

$$-N = \frac{P^2}{4} + \frac{(H_1 - H_2)^2}{P^2} - (H_1 + H_2) \quad (4.9)$$

Let us assume that  $\Psi$  is also eigenstate of the linear momentum, that is  $P^{\alpha} \Psi = k^{\alpha} \Psi$ , where  $k$  is timelike.

Since  $P^2$  and  $y \cdot P$  are simultaneously diagonalized, so are the quantities  $P \cdot p_1$  and  $P \cdot p_2$ . It is usually required that the eigenvalues of these operators are both positive, which amounts to say

$$\frac{1}{2} k^2 > |\nu| \quad (4.10)$$

Justification for this condition is easy when the mutual interaction is  $V = f(Z, P^2)$ . In the underlying classical theory [15] the equations of motion imply that  $P \cdot z = P \cdot p_1 \tau_1 - P \cdot p_2 \tau_2 + const.$  where  $\tau_1, \tau_2$  are the evolution parameters (generalizations of the proper times). On the equal-time surface  $(\Sigma) \simeq P \cdot z = 0$  we get the relation  $P \cdot p_1 \tau_1 = P \cdot p_2 \tau_2 + const.$  between the parameters. It is clear that  $\tau_1$  must increase together with  $\tau_2$ , which implies that *classically*  $P \cdot p_1$  and  $P \cdot p_2$  have the same sign (necessarily positive, since their sum is  $P^2$ ). Equation (4.10) is just the quantum counterpart of this condition. Extension of this argument to more general interactions is an open question, but other considerations motivated by the Bethe-Salpeter equation and propagator theory [16] strongly support condition (4.10).

We can obviously write

$$\Psi = \exp(ik \cdot Q)\psi(z) \quad (4.11)$$

Using also (4.4) we cast eq. (4.8) into the form

$$\left(\frac{k^2}{4} + \frac{\nu^2}{k^2} + N\right)\Psi = \mu\Psi \quad (4.12)$$

Let us set

$$\lambda = \frac{k^2}{4} + \frac{\nu^2}{k^2} - \mu \quad (4.13)$$

(4.12) now reads

$$(N + \lambda)\Psi = 0 \quad (4.14)$$

which suggests to consider (4.14) as an eigenvalue problem where  $\lambda$  is the eigenvalue for the operator  $-N$ . Caution is needed however, in the cases where  $V$  actually depends on  $P^2$  (*energy-dependent case*); in this case  $k^2$  may be substituted to  $P^2$  into the potential and further eliminated from (4.14) with help of (4.13) (as we shall see there is only one way to solve (4.13) for  $k^2$ ). But one is left with an extra dependence on  $\lambda$  in equation (4.14). This complication, first pointed out by Rizov, Sazdjian and Todorov [12], will be discussed below, in the context of the reduced problem.

Indeed equation (4.4) implies that the form (4.11) of the wave function can be further reduced. Recall that  $y_\alpha = -i\partial/\partial z^\alpha$ . In a frame adapted to  $k$ , we can write  $y \cdot k = y_0 k^0$  thus (4.4) entails  $\psi = \exp(i\nu z^0/k^0) \phi$  where  $\phi$  does not depend on  $z^0$ . In fact  $\phi$  belongs to a linear space which depends on the direction of the vector  $k^\alpha$ , but *does not depend* on the number  $k = \sqrt{k^\alpha k_\alpha}$ . Imposing square integrability defines a Hilbert space, say  $L^2(k^\alpha)$ . Now eq. (4.8) or (4.12) becomes an equation to be solved in  $L^2(k^\alpha)$  because after substitution of  $k$  for  $P$  and  $\nu$  for  $(y \cdot P)$  in  $N$  we obtain an operator acting in  $L^2(k^\alpha)$ , say  $(N)_{\nu, k}$ . Insofar as  $V$  depends on  $P^2$ , the operator  $(N)_{\nu, k}$  depends on the scalar  $k^2$ . We can solve (4.13) for  $k$  and substitute the result in  $(N)_{\nu, k}$ . In other words, we insert

$$k^2 = 2(\lambda + \mu) + 2\sqrt{(\lambda + \mu)^2 - \nu^2} \quad (4.15)$$

into  $(N)_{\nu, k}$ .

Notice that an alternative solution of (4.13) corresponding to the minus sign in front of the radical is discarded because it would violate the condition  $\frac{1}{2}k^2 > |\nu|$ . In contradistinction, provided that  $\lambda + \mu > \nu$  we are sure that this condition is satisfied by solution (4.15). As a result the correspondence between  $\lambda$  and  $k^2$  remains one to one for unequal masses.

But, for energy-dependent interactions, we end up with a reduced wave equation which fails to be an eigenvalue equation in the usual sense, being nonlinear in the eigenvalue  $\lambda$  [17]. This complication, pointed out in ref.[12] stems from the possible dependence of the mutual interaction term  $V$  on the (squared) total energy  $P^2$ .

Fortunately, a mathematical theory exists for generalized eigenvalue equations which are nonlinear in the eigenvalue [18]. But this theory involves technical sophistications resorting to the concept of "associated vectors". Its main departure from the standard theory is that eigenvectors corresponding to different generalized eigenvalues may fail to be orthogonal. This drawback (not characteristic of the two-body problem, see [19]) led the authors of ref. [12] to advocate a re-definition of the scalar product of eigenvectors. In this paper we shall focus on situations where this redefinition is unnecessary.

No such problem arises when  $V$  does not depend on the energy. Any potential of the form  $V = f(Z/P^2, y \cdot P)$  (in particular the harmonic oscillator  $V = \text{const.} z^2$ ) leads, in absence of external field, to an ordinary eigenvalue equation in  $L^2(k^\alpha) = L^2(\mathbf{R}^3, d^3 z_\perp)$ , where  $z_\perp$  is orthogonal to  $k$ .

This class of potential encompasses a lot of phenomenological models. In contrast, realistic potentials motivated by QED pertain to the other type.

#### 4.2 The reduced equation in the presence of external field.

Let us now return to the case where an external potential is applied to our system. The aim of this section is only to reduce the wave equations. The discussion of all matters concerning the spectrum are postponed to Section 5.

Since  $C$  is the natural generalization of linear momentum in the presence of external field, and by analogy with (4.9), we shall *define*

$$-N = \frac{C^2}{4} + \frac{(H_1 - H_2)^2}{C^2} - (H_1 + H_2) \quad (4.16)$$

Although formula (4.7) is no longer valid, the sum of the basic wave equations still reads  $(N + \lambda)\Psi = 0$  with  $\lambda$  defined as in (4.13). We assume that  $\Psi$  is eigenstate of the pseudomomentum, say  $C^\alpha \Psi = k^\alpha \Psi$ , for some constant  $k$  which is supposed to be strictly *timelike*. By analogy with the isolated system and also for pragmatic reasons, we shall confine ourselves to the sector  $k^2/2 > |\nu|$ . After transformation to the convenient representation we get

$$-N'' = \frac{P^2}{4} + \frac{(H_1'' - H_2'')^2}{P^2} - (H_1'' + H_2'') \quad (4.17)$$

and our goal is to determine the spectrum of this quantity. Then one will pass from  $\lambda$  to  $k^2$  through (4.13).

The calculations can be organized as follows:

Whereas (2.22) fixes the dependence in the relative time, eq.(2.23) allows us to factorize out the "center-of-mass motion", and we are left with the reduced wave function  $\phi$  which arises in eq. (2.23). Obviously (2.22) implies that

$$y_L \cdot k_L \phi = 0 \quad (4.18)$$

thus  $\phi$  depends on  $z$  only through its projection  $\varpi z$ . Imposing square integrability amounts to require that  $\phi$  is in the Hilbert space  $L_k^2$  defined by (2.24). It is clear that  $\phi$  generally depends on  $\nu$  and  $k$  as parameters.

In order to write down the reduced wave equation, we replace  $P^\alpha$  and  $y_L \cdot P_L$  respectively by their eigenvalues  $k^\alpha$  and  $\nu$  in  $H_1'' + H_2''$ , and we divide by exponential factors.

For any operator  $\mathcal{O}$  it is convenient to use the following convention

$$(\mathcal{O})_{\nu, k} = \mathcal{O}|_{y_L \cdot P_L = \nu, P = k} \quad (4.19)$$

The subscript  $k$  refers to the vector  $k$ , which finally contributes by its longitudinal piece only. Notice that here a term like  $y^2$  must be written as  $y^2 \equiv (\Omega y)^2 + \frac{(y_L \cdot P_L)^2}{P_L^2}$ . If we now introduce the projector  $\varpi$  orthogonal to  $k_L$ , and use the identity  $\frac{1}{k_L^2} = \frac{1}{k^2}(1 - \epsilon^2)$  we obtain

$$\left(\frac{P^2}{4} + y^2\right)_{\nu, k} = \frac{k^2}{4} + (\varpi y)^2 + \frac{\nu^2}{k_L^2} = \frac{k^2}{4} + (\varpi y)^2 + \frac{\nu^2}{k^2} - \epsilon^2 \frac{\nu^2}{k^2} \quad (4.20)$$

which is to take into account when computing  $(K_1'' + K_2'')_{\nu, k}$  from (3.14).

Naturally  $(H_1'' + H_2'')_{\nu, k} = (K_1'' + K_2'')_{\nu, k} + 2(V'')_{\nu, k}$ . Defining

$$R(\nu, k_L, k_T) = (K_1'' + K_2'')_{\nu, k} \quad (4.21)$$

$$W(\nu, k_L, k_T) = (V'')_{\nu, k} \quad (4.24)$$

we end up with the equation

$$R \phi + 2W \phi = \mu \phi \quad (4.23)$$

At this stage, let us observe that, in agreement with a remark made in Section 2, it is expected that neither  $R$  nor  $W$  involve  $z_L \cdot P_L$ . This will be checked later on and will permit us to consider  $R$  and  $W$  as operators acting in the Hilbert space  $L^2(k_L)$  defined by (2.24).

We remind that  $\mu$  is just a parameter fixed from the outset. The question whether (4.23) can be considered as a spectral problem, *and for which eigenvalue*, will be considered later on, with help of equations (4.13)(4.16). See eq. (4.33) below.

The explicit expression of  $R$  comes from (3.14), with help of (4.21), and at first sight seems very involved. This apparent complication results from having used the necessary transformations (2.4)(2.19). Things become more tractable if we separate all terms involving  $F^{\mu\nu}$  from those which survive when the field is turned off, and so on. So let us perform such a separation in a systematic way. Be cautioned that  $(\mathcal{O}'')^{(0)} \neq (\mathcal{O}^{(0)})''$ .

Since  $K_1'', K_2''$  are no more than quadratic in the field strength, we can write

$$K_1'' + K_2'' = (K_1'' + K_2'')^{(0)} + (K_1'' + K_2'')^{(1)} + (K_1'' + K_2'')^{(2)} \quad (4.24)$$

with the obvious convention that superscripts (1), (2) respectively refer to (homogeneous) linear and quadratic terms in the field strength. Therefore  $R = R^{(0)} + R^{(1)} + R^{(2)}$ . In eq.(3.14) we have to replace  $P$  by  $k$  and  $y_L \cdot P_L$  by  $\nu$ , in order to compute  $R$ . Remembering (2.17) we start from (3.14) and compute  $K_1'' + K_2''$  (to be inserted into (3.1) and further simplified by the above substitution). We first consider the piece of it which survives in the no-field limit. In eq. (3.14) we collect zeroth order terms in the field and find

$$(K_1'' + K_2'')^{(0)} = \frac{P^2}{4} + \frac{(y_L \cdot P_L)^2}{P_L^2} + (\Omega y)^2 + y_T \cdot P_T \frac{y_T \cdot P_T - 2y_L \cdot P_L}{P_L^2} \quad (4.25)$$

According to notation (4.19) we can write

$$(K_1'' + K_2'')_{\nu,k}^{(0)} = \frac{k^2}{4} + \frac{\nu^2}{k_L^2} + (\varpi y)^2 + y_T \cdot k_T \frac{y_T \cdot k_T - 2\nu}{k_L^2} \quad (4.26)$$

At this stage, one might be tempted to separate out the first two terms in the right-hand-side because they are constant. However the second one depends on  $k_L$  which loses intrinsic meaning in the no-field limit. Therefore we prefer to apply the identity  $\frac{1}{k_L^2} = \frac{1}{k^2}(1 - \epsilon^2)$  which yields

$$(K_1'' + K_2'')_{\nu,k}^{(0)} = \frac{k^2}{4} + \frac{\nu^2}{k^2} - \epsilon^2 \frac{\nu^2}{k^2} + (\varpi y)^2 + \frac{(y_T \cdot k_T)^2}{k_L^2} - 2\nu \frac{y_T \cdot k_T}{k_L^2} \quad (4.27)$$

Now setting

$$(S)_{\nu,k} = (\varpi y)^2 + y_T \cdot k_T \frac{y_T \cdot k_T - 2\nu}{k_L^2} - \epsilon^2 \frac{\nu^2}{k^2} \quad (4.28)$$

we can write

$$(K_1'' + K_2'')_{\nu,k}^{(0)} = \frac{k^2}{4} + \frac{\nu^2}{k^2} + (S)_{\nu,k} \quad (4.29)$$

so we end up with

$$R^{(0)} = \frac{k^2}{4} + \frac{\nu^2}{k^2} + (S)_{\nu,k} \quad (4.30)$$

The field-depending terms in (3.14) provide

$$R^{(1)} = 4e(z \cdot F \cdot y) \frac{\nu}{k_L^2} - \frac{e}{2} z \cdot F \cdot k \quad (4.31)$$

$$R^{(2)} = \frac{e^2}{4}(z \cdot F)^2 + 4e^2 \frac{(z \cdot F \cdot y)^2}{k_L^2} \quad (4.32)$$

Remember that  $F$  has transverse components only. Contractions involving  $F$  only depend on the transverse components; for instance  $F \cdot k$  is just a combination of the quantities  $k_T^\alpha$ . It is noteworthy that only the transverse components of  $z, y$  arise in  $R^{(1)}, R^{(2)}$ , whereas  $(S)_{\nu,k}$  depends on  $\varpi y$  and  $y_T$ . As a whole,  $R$  depend only on  $\varpi z$  and  $\varpi y$  (recall  $y_T, z_T$  are pieces of  $\varpi y, \varpi z$  respectively).

Let us again define  $\lambda$  by formula (4.13). In view of (4.30)(4.31)(4.32), equation (4.23) may be written

$$\lambda\phi + [(S)_{\nu,k} + R^{(1)} + R^{(2)} + 2W]\phi = 0 \quad (4.33)$$

The square bracket in (4.33) is nothing but  $(-N'')_{\nu,k}$ .

Remark:  $R^{(0)}, R^{(1)}R^{(2)}$  do not involve the mutual interaction. In contradistinction  $W$  is defined through (4.24) and is model dependent in this sense that it crucially depends on the form of the function  $f$  which determines the mutual interaction.

Owing to the abundance of terms depending on  $\epsilon$  we distinguish the "motional" case, where  $\epsilon \neq 0$ , from the case "at rest" characterized by the vanishing of  $\epsilon$  (or equivalently of  $k_T$ ). Notice that  $R^{(2)}$  does not depend on  $k_T$ . It has the same form in motional case and in the case at rest [20].

Let us evaluate  $W$ . In view of (4.24) we have first to write down the expression for  $V''$ , say (3.15). It follows that

$$W = f((\hat{Z}^\sharp)_{\nu,k}, (P^\sharp)_{\nu,k}^2, \nu) \quad (4.34)$$

In this formula  $(P^\sharp)^2$  is given by (3.6) and  $\hat{Z}^\sharp$  by (3.16). Making the substitutions  $P \rightarrow k$  and  $y_L \cdot P_L \rightarrow \nu$  we obtain

$$(\hat{Z}^\sharp)_{\nu,k} = (P^\sharp)_{\nu,k}^2 (\varpi z)^2 - (z_T \cdot k)^2 \quad (4.35)$$

$$(P^\sharp)^2_{\nu,k} = k^2 + e k \cdot F \cdot z + \frac{e^2}{4} (F \cdot z)^2 \quad (4.36)$$

It is clear that  $W$  does not involve the operator  $z \cdot k_L$ . Formulas (4.35)(4.36) are to be inserted into (4.34), then the explicit form of  $W$  will come out. This last expression, together with (4.30)(4.31)(4.32) provides the explicit form of the eigenvalue equation (4.33), where  $\lambda$  is the eigenvalue of an operator  $-(N)_{\nu,k}$  acting in  $L^2(k_L)$ .

Still we meet a complication: just like in the case of an isolated system, we can solve (4.13) for  $k^2$  (condition (4.10) ensures that relation (4.13) can be uniquely inverted) and insert the result into  $(N'')_{\nu,k}$ . Similarly  $k_L^2$  can be replaced by  $\frac{k^2}{1 - \epsilon^2}$ . As a result eq. (4.33) bears an extra dependence on  $\lambda$ . In general it is a *nonconventional eigenvalue equation*.

Naturally a scalar product for the reduced wave function has to be explicitly defined. This question will be discussed in next section, in analogy with the line we have followed in the case of an isolated system.

According to relation (4.13), a discrete spectrum for  $\lambda$  would imply that the admissible values of  $k^2$  are restricted to a discrete sequence.

### 4.3. Discussion

Finally the system (2.21)(2.22) has been reduced to the three-dimensional problem of solving (4.33). This formula is nonlinear in the field strength and may be applied to strong fields. Let us analyze the various contributions it contains. We distinguish motional terms, depending on  $\epsilon$  or  $k_T$ . In fact we can write  $k_T = \epsilon \Lambda k_L$  where the second rank tensor  $\Lambda$  represent the boost from the direction of  $k_L$  to the direction of  $k_T$  (thus  $\Lambda \cdot \Lambda = \delta$ ).

Loosely speaking we could say that, in as much as the shape of  $W$  departs from the original form assumed by  $V^{(0)}$ , the mutual interaction is "modified by the magnetic field".

#### a) system at rest

The particular case where pseudomomentum is purely longitudinal enjoys a particular simplicity. If we assume for a moment that  $k$  coincides with  $k_L$ , it is possible to find a frame where  $\mathbf{k}$  vanishes whereas the electromagnetic field is purely magnetic. In other words the pseudo rest frame is also a lab frame. We refer to this situation as the case *at rest*.

In this case,  $\varpi z = z_\perp$ ,  $\varpi y = y_\perp$  and  $(S)_{\nu,k}$  simply reduces to  $y_\perp^2$ . Naturally  $L^2(k_L)$  coincides with  $L^2(k)$ . We notice that the second term in the r.h.s. of (4.35) vanishes, hence  $(\widehat{Z}^\sharp)/P^{\sharp 2})_{\nu,k}$  reduces to  $z_\perp^2$ . Thus, for energy independent interactions ( $V^{(0)} = g(Z/P^2, y \cdot P)$ ),  $W$  assumes the form  $g(z_\perp^2, \nu)$ . In other words: *At rest, the magnetic field does not modify the mutual interaction when it is not energy-dependent.*

All surviving terms in (4.33) can easily be identified as covariant generalizations of the usual terms present in the non-relativistic theory, except for a piece of  $R^{(2)}$  which depends on the relative angular momentum (formula (4.32)) and remains small for heavy bound states; its contribution however might be significant for light bound states ( $k^2$  small) in a strong magnetic field.

At first order in the field strength, the relative motion admits no correction other than a term proportional to  $\nu$  (indeed  $F \cdot k$  vanishes). For equal masses there is no departure from the motion of an isolated system.

### b) motional case

When  $k_T$  is nonzero, we recognize the motional electric field contained in  $F \cdot k$ .

Energy-dependent interactions seem to be more sensitive to the external field. As can be read off from (4.36), the modification implied by (4.34) is nonlinear in  $F$ . This point may become important in strong fields.

Even if the interaction is independent of the energy, the presence of the motional term  $z_T \cdot k$  in (4.35) entails a non-trivial difference between (4.34) and (2.2).

## 5. NORMALIZATION AND PERTURBATION THEORY.

### 5.1. Normalization

In connexion with the rise of a reduced wave function  $\phi$  in (2.23), we are led to consider in general any function (or possibly a distribution) of  $z$  of the form  $\phi(k, z)$  which depends on  $z$  only through the combination  $\varpi z$  (assuming  $k$  timelike), regarding the four-vector  $k$  as a given parameter. The most straightforward normalization of  $\phi$  is given by a three-fold integral using the volume element  $d^3(\varpi z)$ , with the convention that  $\varpi z = z$  in any frame adapted to  $k_L$ . In other words, for each  $k$ , we separate  $k_L$  from  $k$  and consider the Hilbert space  $L^2(k_L) = L^2(\mathbf{R}^3, d^3(\varpi z))$ .

For a function of the above type, but not necessarily solution to (4.33), we have, with an obvious notation

$$\langle \phi, \phi \rangle_k = \int \phi^* \phi \, d^3(\varpi z) \quad (5.1)$$

Here  $\phi$  may additionally depend on  $\nu$  and  $k_T$  as parameters. The label  $k$  refers to the *vector*  $k$ , but actually  $L^2(k_L)$  depends on  $k^\alpha$  only through the direction of  $k_L$ .

It is noteworthy that the normalization (5.1) of reduced wave functions is not only dictated by simplicity, but also consistent with the *off-shell normalization* of the full wave functions.

The proof of this point will be given elsewhere; its precise meaning is as follows:

Let us consider any function  $\Psi''(Q, z)$  irrespective of its being on the mass shell or not.

Assuming that  $\phi(z^\alpha, k^\beta, \nu)$  is related to  $\Psi''$  through (2.23), it is possible to check that

$$\int \Psi''^* \Psi'' \, d^4 Q \, d^4 z = \int \langle \phi, \phi \rangle_k \, d^4 k \, d\nu$$

provided that the Fourier transform of  $\Psi''$  with respect to  $Q$ , say  $\Xi(k, z)$ , is a retarded function of  $k$  (it vanishes outside the region limited by the positive sheet of the light cone).

Of course this nice property [21] does not prevent the complications associated with energy dependence.

Even in the simple case where the mutual interaction  $V^{(0)}$  (present in the isolated system) does not depend on  $P^2$ , we are bound to realize that the spectral problem is not a standard one, due to the presence of the magnetic field. This can be seen as follows. First we notice that the occurrence of  $(\widehat{Z}^\sharp)_{\nu,k}$  in  $W$  brings out a dependence on  $k^2, k_L^2$ . Second we observe an unescapable dependence on  $k^2$  and  $k_L^2$  in formulas (4.28)(4.31)(4.32). Clearly, we cannot expect a conventional spectral problem. This peculiarity arises because going from  $\Psi$  to  $\Psi''$ , we have abandoned the customary representation; transformation formulas (2.4)(2.19) are responsible for the nonlinear dependence on  $\lambda$ .

## 5.2 Outline of a perturbative approach.

In order to solve the reduced wave equation, perturbation theory is by no means straightforward. The abundant mathematical literature devoted to nonlinear eigenvalue equations [18] pays very little attention to spectrum perturbation. So the general case seems to be hopeless in the present state of the art [22].

*Henceforth we limit ourselves to the simple case where  $V^{(0)}$  is not energy dependent.* Still we must cope with a nonconventional eigenvalue equation when magnetic field is present. But we can make a couple of important remarks.

First we observe that all terms involving energy dependence vanish when *both* the external field and the motional parameter vanish.

Second, we notice that  $\epsilon$  and  $k^2$  are two independent parameters contributing to the determination of the vector  $k^\alpha$ . Inspection of (4.13) shows that  $\epsilon$  and  $\lambda$  are mutually independent.

These remarks open the possibility to *treat simultaneously electromagnetic contributions and motional terms as a perturbation*. This procedure requires that the motional parameter is not too large.

In this approach the unperturbed equation is free of nonlinear dependence on  $\lambda$ . It corresponds to "static states" of the isolated system, *i.e.* states with external coupling removed and additionally at rest with respect to the special lab frame; they are characterized by  $k_T = 0$ ,  $F = 0$  (see Appendix). Only the perturbation, which combines motional terms and field contributions, is affected by energy dependence. It is fortunate that Rizov, Sazdjian and Todorov [12] have developed a perturbative scheme (tractable by physicists) for this situation.

Another departure from conventional perturbation theory lies in the fact that the perturbation is nonlinear in  $\epsilon$  and in the field strength.

Indeed, insofar as  $f$  in (2.6) can be developed as an analytic function of its arguments, a look at (4.35)(4.36) indicates that even if  $f$  is linear in  $Z$  (harmonic potential),  $V''$  and  $W$  are at least quadratic in the external field.

Fortunately, the formalism set up in [12] is general enough to accomodate second (and higher) order perturbations.

The first step is a *weak-field-slow-motion approximation*. The field strength and  $\epsilon$  are taken into account only at first order. The perturbation scheme of ref. [12] gets simplified; it turns out that the correction to the eigenvalue is given by the usual formulas. In particular if the unperturbed eigenvalue is nondegenerate, the correction is still given by the expectation value of the perturbation term in the eigenstate of the unperturbed problem. We can organize calculations by setting  $F/\mu = \epsilon G$  (or alternatively  $F/k^2 = \epsilon G$ ) where the tensor  $G$  is dimensionless. It follows that  $F \cdot k$  and the motional electric field are  $O(\epsilon^2)$ . The first order treatment entails enormous simplifications. For instance the obligation to distinguish  $(z_\perp)^2$  from  $(\varpi z)^2$  is in general a serious computational complication, but these quantities coincide at first order in  $\epsilon$ . A glance at (4.35) shows that  $W = f(z_\perp^2, \nu) + O(\epsilon^2)$ . At first order, the only departure from the unperturbed equation consists in the  $\nu$ -terms and vanishes for equal masses. To summarize:

*If the neutral system as a whole undergoes a slow motion in a weak field, the relative motion is affected by the magnetic field only by a contribution which vanishes for equal masses.*

Therefore it is necessary to go beyond first order in search for a nonzero correction.

*Example:*

Suppose that  $V^{(0)} = g(\tilde{z}^2)$ . This form of mutual interaction includes the relativistic harmonic oscillator, say  $V^{(0)} = \gamma Z/P^2$ , where  $\gamma$  is a coupling constant. The only difference from the unperturbed equation comes from  $R^{(1)}$ , which is given by (4.31).

Another situation of great simplicity consists in a system of equal masses at rest ( $\epsilon$  and  $\nu$  are strictly zero). The perturbation reduces to  $R^{(2)}$ , hence is linear in  $|F|^2$ . Corrections to the levels are easy to calculate, but significant only in strong fields.

## 6. CONCLUSION.

The coupled Klein-Gordon equations describing a neutral system have been reduced to a three-dimensional eigenvalue equation involving truly motional effects and recoil effects in a covariant fashion. Moreover the particular symmetry associated with a constant magnetic field *in space-time* is manifestly respected.

Explicit formulas are written in a representation which permits to easily satisfy compatibility and also to eliminate relative time and the  $Q$ -variables. The surviving degrees of freedom are the same as in the nonrelativistic theory.

In the reduction procedure it was essential to consider eigenstates of pseudomomentum. The square of this vector plays the role of an effective squared mass which can be, in principle, evaluated by solving the reduced equation. Indeed (in the sector we have considered, and especially for equal masses) the eigenvalue  $\lambda$  is in one-to-one correspondence with  $k^2$ . Bound states are characterized by  $\lambda$  in the discrete spectrum.

In the most natural way, our approach generalizes to the magnetic case the usual treatment of isolated two-body systems according to "predictive mechanics" and "constraints theory" [6][7][16][8]. It provides a clean theoretical basis for the study of neutral bound states in the relativistic regime.

The equations of motion are nonlinear in the field strength and offer a starting point for investigation of strong field effects. In principle, they encompass all kinematic possibilities of the system as a whole and permit a description of ultra-relativistic situations. The Ansatz which allows for a three-dimensional reduction in the covariant framework automatically generates various terms in the wave equation. From a practical point of view, some of them play the role of a modification of the mutual interaction, caused by the magnetic field. This effect might have dramatical consequences in a system where the mutual interaction is energy-depending (for instance Coulombian interaction along the lines of the quasi-potential approach). But in this case we cannot go beyond qualitative estimations, since the perturbative scheme borrowed from ref. [12] is limited to the systems where the interaction is not energy dependent. This point may be a motivation for eventually undertaking a more general perturbation theory applicable also to eigenvalue equations involving an energy-dependent potential.

When  $V^{(0)}$  does not depend on  $P^2$ , spectrum calculations can be handled in a perturbative approach where the motional parameter as well as the field strength must necessarily be small. For the moment this method can be used for simple and idealized systems bounded by a phenomenological potential. Application to the relativistic "naive quark" model of hadron seems to indicate that the spectrum enjoys some kind of stability with respect to magnetic perturbation. Of course the polarizability of this model can be studied in our formalism.

In the hope of obtaining a nonvanishing correction due to the presence of external field, we are obliged to go beyond first order. In a future work we plan to investigate with more details the case  $\epsilon = \nu = 0$  sketched in last section.

More work is needed in search for ultra relativistic effects ( $\epsilon \simeq 1$ ). A less ambitious program might be the computation of second order motional effects in the framework of the perturbative scheme considered here. But the complication of the calculations may be prohibitive.

Naturally an extension to particles with spin is desirable.

Let us finally mention that the contact with usual methods of quantum field theory could be improved by considering a Bethe-Salpeter equation taking the magnetic field into account from the start. This would mean to resume the work of Bijtebier and Broeckaert [23] in a way which respects the particular symmetry of constant magnetic field, *i.e.* treating all the possible lab frames on the same footing.

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## APPENDIX

### The unperturbed equation

The vanishing of  $k_T$  (equivalently of  $\epsilon$ ) makes  $L^2(k_L)$  to coincide with  $L^2(k)$  and  $\varpi z$  with  $z_\perp$ . As also  $F$  is zero, inspection of (4.34)(4.35)(4.36) shows that  $W$  reduces to  $f((Z)_{\nu,k}, k^2, \nu) = f(z^2 k^2 - (z \cdot k)^2, k^2, \nu)$ . Finally we see that, for  $\epsilon = F = 0$ , (4.33) reduces to

$$\lambda\phi + z_\perp^2\phi + f(z^2 k^2 - (z \cdot k)^2, k^2, \nu)\phi = 0$$

It is exactly the equation one would obtain for the isolated system after reduction, in the original representation.

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